CCGPS Frameworks Student Edition

Mathematics

CCGPS Analytic Geometry
Unit 2: Right Triangle Trigonometry

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"Making Education Work for All Georgians"

Unit 2
Right Triangle Trigonometry

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OVERVIEW

In this unit students will:

- explore the relationships that exist between sides and angles of right triangles.
- build upon their previous knowledge of similar triangles and of the Pythagorean Theorem to determine the side length ratios in special right triangles
- understand the conceptual basis for the functional ratios sine and cosine
- explore how the values of these trigonometric functions relate in complementary angles
- to use trigonometric ratios to solve problems.
- develop the skills and understanding needed for the study of many technical areas
- build a strong foundation for future study of trigonometric functions of real numbers.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Define trigonometric ratios and solve problems involving right triangles.

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

RELATED STANDARDS
Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression

of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the

problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the

Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Similar right triangles produce trigonometric ratios.
- Trigonometric ratios are dependent only on angle measure.
- Trigonometric ratios can be used to solve application problems involving right triangles.

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers, integers and irrational numbers, including application of order of operations
- operations with algebraic expressions
- simplification of radicals
- basic geometric constructions
- properties of parallel and perpendicular lines
- applications of Pythagorean Theorem
- properties of triangles, quadrilaterals, and other polygons
- ratios and properties of similar figures
- properties of triangles

SELECTED TERMS AND SYMBOLS

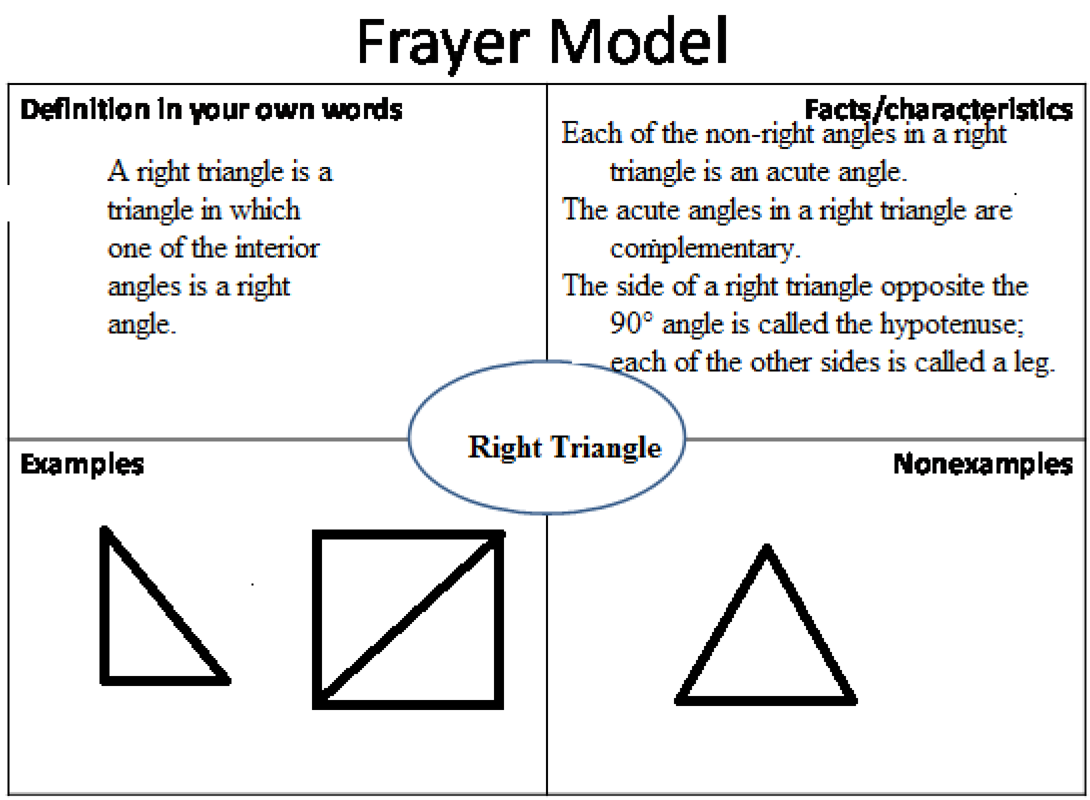
According to Dr. Paul J. Riccomini, Associate Professor at Penn State University,

"When vocabulary is not made a regular part of math class, we are indirectly saying it ***isn't important!****"* (Riccomini, 2008) Mathematical vocabulary can have significant positive and/or negative impact on students' mathematical performance.

- Require students to use mathematically correct terms.
- Teachers must use mathematically correct terms.
- Classroom tests must regularly include math vocabulary.
- Instructional time must be devoted to mathematical vocabulary.

http://www.nasd.k12.pa.us/pubs/SpecialED/PDEConference//Handout%20Riccomini%20Enhancing%20Math%20InstructionPP.pdf

For help in teaching vocabulary, a Frayer model can be used. The following is an example of a term from earlier grades. http://wvde.state.wv.us/strategybank/FrayerModel.html



More explanations and examples can be found at
http://oame.on.ca/main/files/thinklit/FrayerModel.pdf

The following terms and symbols are often misunderstood. Students should explore these concepts using models and real-life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

- **Adjacent side:** In a right triangle, for each acute angle in the interior of the triangle, one ray forming the acute angle contains one of the legs of the triangle and the other ray contains the hypotenuse. This leg on one ray forming the angle is called the adjacent side of the acute angle.

For any acute angle in a right triangle, we denote the measure of the angle by θ and define three numbers related to θ as follows:

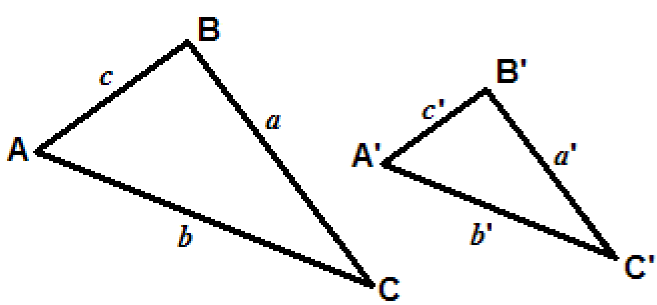
$$\textbf{sine of } \theta = \sin(\theta) = \frac{\textbf{length of opposite side}}{\textbf{length of hypotenuse}}$$

$$\textbf{cosine of } \theta = \cos(\theta) = \frac{\textbf{length of adjacent side}}{\textbf{length of hypotenuse}}$$

- **Complementary angles:** Two angles whose sum is 90° are called **complementary.** Each angle is called the complement of the other.

- **Opposite side:** In a right triangle, the side of the triangle opposite the vertex of an acute angle is called the opposite side relative to that acute angle.

- **Similar triangles:** Triangles are similar if they have the same shape but not necessarily the same size.
  - Triangles whose corresponding angles are congruent are similar.
  - Corresponding sides of similar triangles are all in the same proportion.
  - Thus, for the similar triangles shown at the right with angles A, B, and C congruent to angles A**'**, B**'**, and C**'** respectively, we have that:, we have that:



$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Properties, theorems, and corollaries:

- For the **similar triangles**, as shown above, with angles A, B, and C congruent to angles A**'**, B**'**, and C**'** respectively, the following proportions follow from the proportion between the triangles.

  $\frac{a}{a'} = \frac{b}{b'}$ if and only if $\frac{a}{b} = \frac{a'}{b'}$; $\frac{a}{a'} = \frac{c}{c'}$ if and only if $\frac{a}{c} = \frac{a'}{c'}$;

  and $\frac{b}{b'} = \frac{c}{c'}$ if and only if $\frac{b}{c} = \frac{b'}{c'}$.

  Three separate equalities are required for these equalities of ratios of side lengths in one triangle to the corresponding ratio of side lengths in the similar triangle because, in general, these are three different ratios. The general statement is that the ratio of the lengths of two sides of a triangle is the same as the ratio of the corresponding sides of any similar triangle.

- For each pair of complementary angles in a right triangle, the sine of one angle is the cosine of its complement.

This web site has activities to help students more fully understand and retain new vocabulary (i.e. the definition page for *dice* actually generates rolls of the dice and gives students an opportunity to add them).

http://www.teachers.ash.org.au/jeather/maths/dictionary.html

Definitions and activities for these and other terms can be found on the Intermath website http://intermath.coe.uga.edu/dictnary/homepg.asp.

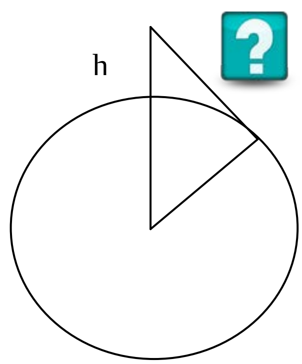
Horizons

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Modified from NCTM's On Top of the World
http://illuminations.nctm.org/LessonDetail.aspx?id=L711

Have you ever visited the capitol of Georgia: Atlanta? If you have, you may have seen the tallest structure in Atlanta; the Bank of America Plaza is 1,023 feet tall, making the building the 9th tallest building in the country.

If you could stand on the top of this building, how far is the horizon? In other words, how far could you see? This distance can be calculated by using right triangles and knowing that the radius of the earth is approximately 3963 miles.



Some preliminary data:

- The angle formed by the radius of a circle and a tangent line to the circle is a right angle.
- 3963 miles converted to feet is 3963 miles x 5280 feet/mile = 20,924,640 feet.

If h represents the height of the plaza, 1,023 feet, then the hypotenuse of the triangle is 1023 + 20,924,640 = 20,925,663 feet.

Setting up the Pythagorean Theorem would be $20{,}925{,}663^2 = 20{,}924{,}640^2 + ?^2$. So, if you could stand on the top of Atlanta's tallest building, the distance to the horizon would be approximately 206,913 feet or around 39 miles.

Your assignment is to find the distance to the horizon if you are standing on top of

1. Another building in Atlanta.
2. A building in your home city.
3. A building in another part of the United States.
4. A building in another country.

Eratosthenes Finds the Circumference of the Earth Learning Task

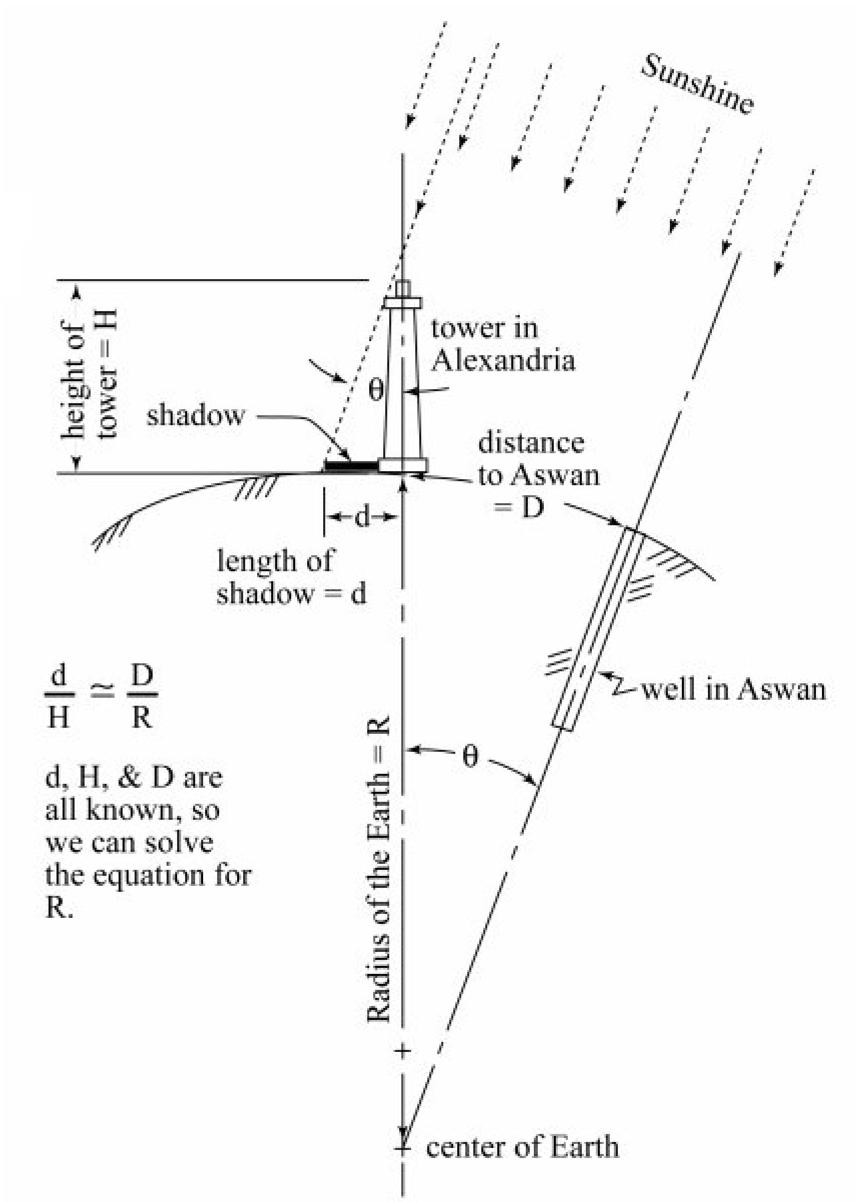
MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

As Carl Sagan says in the television series Cosmos: Two complex ideas, the wheel and the globe, are grooved into our minds from infancy. It was only 5500 years ago that we finally saw how a rotating wheel could produce forward motion. Recognizing that Earth's apparently flat surface bends into the shape of a sphere was even more recent. Some cultures imagined Earth as a disc, some, box-shaped. The Egyptians said it was an egg, guarded at night by the moon. Only 2500 years ago, the Greeks finally decided Earth was a sphere. Plato argued that, since the sphere is a perfect shape, Earth must be spherical. Aristotle used observation. He pointed to the circular shadow Earth casts on the moon during an eclipse.

The Greeks had no way of knowing how large the globe might be. The most daring travelers saw Earth reaching farther still beyond the fringe of their journeys. Then, in 200 BC, travelers told the head of the Alexandria Library, Eratosthenes, about a well near present-day Aswan. The bottom of the well was lit by the sun at noon during the summer solstice. At that moment the sun was straight overhead. Eratosthenes realized he could measure the shadow cast by a tower in Alexandria while no shadow was being cast in Aswan. Then, knowing the distance to Aswan, calculating the Earth’s radius would be simple.

In this task, you will examine the mathematics that Eratosthenes used to make his calculations and explore further the mathematics developed from the relationships he used.

1. Looking at the diagram below, **verify that** the two triangles are similar: the one formed by the sun’s rays, the tower, and its shadow, and the one formed by the sun’s rays, the radius of the earth, and the distance to Aswan (ignore the curvature of the Earth as Eratosthenes did) Explain your reasoning.



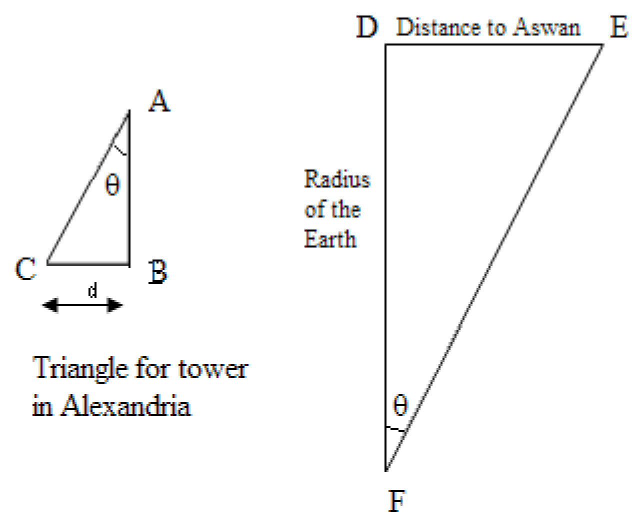
The above diagram is reproduced from the transcript and accompanying diagram for episode 1457 of the radio program The Engines of Our Ingenuity at http://www.uh.edu/engines/epi1457.htm .

2. Focus on the two similar triangles from the diagram in *Item 1*.

a. Write a proportion that shows the relationship of the small triangle to the large triangle.

b. Rearrange the similarity statement in *part a* to match the statement in the given diagram in *Item 1*. Explain why this proportion is a true proportion.

3. Now, look at the triangles isolated from the diagram as shown at the right.



a. Knowing that these triangles represent the original diagram, where should the right angles be located?

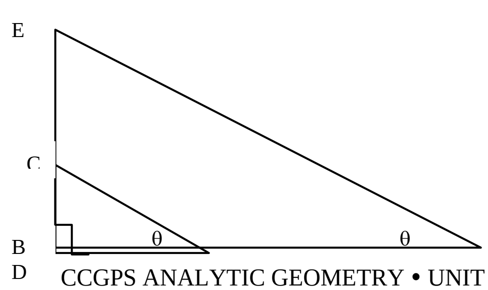
b. Using the right angles that you have identified, identify the legs and hypotenuse of each right triangle.

c. Rewrite your proportion from above using the segments from triangle ABC and triangle DEF.

d. By looking at the triangle ABC, describe how the sides AB and BC are related to angle θ.

e. By looking at the triangle DEF, describe how the sides DE and DF are related to angle θ.

4. If we rearrange the triangles so that the right angles and corresponding line segments align as shown in the figure below, let's look again at the proportions and how they relate to the angles of the triangles.



A F

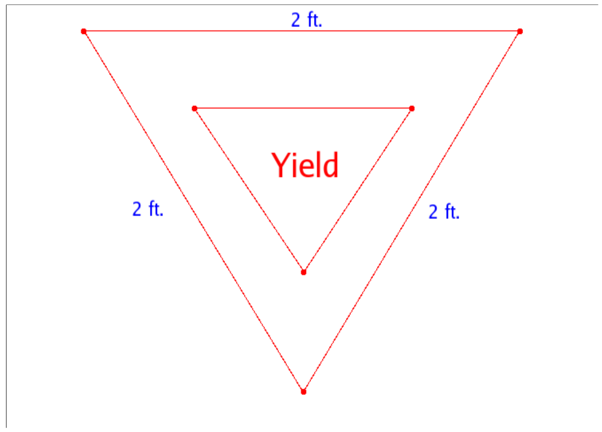
Looking at the proportion you wrote in 3c, $\frac{\overline{BC}}{\overline{AB}} = \frac{\overline{DE}}{\overline{DF}}$ and the answers to 3d and 3e, when would a proportion like this always be true? Is it dependent upon the length of the sides or the angle measure? Do the triangles always have to similar right triangles? Why or why not?

Discovering Special Triangles Learning Task

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

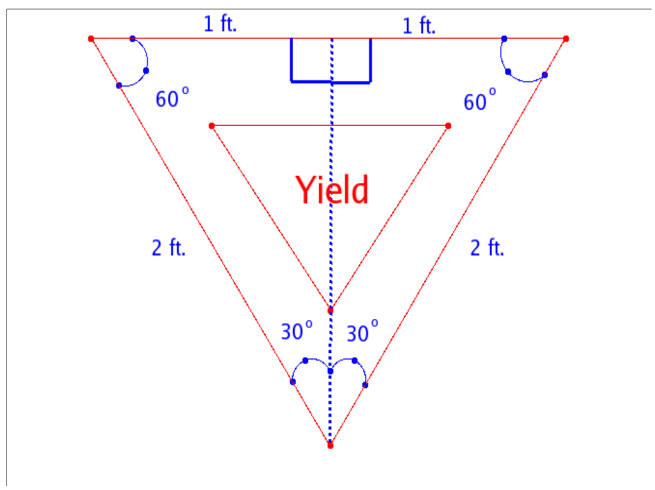
Part 1

1. Adam, a construction manager in a nearby town, needs to check the uniformity of Yield signs around the state and is checking the heights (altitudes) of the Yield signs in your locale. Adam knows that all yield signs have the shape of an equilateral triangle. Why is it sufficient for him to check just the heights (altitudes) of the signs to verify uniformity?

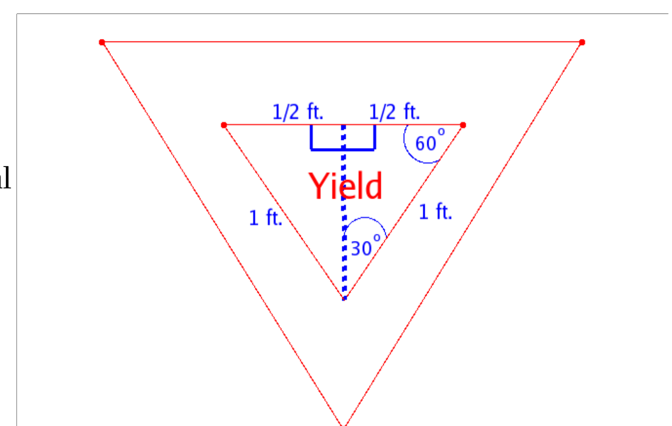


2. A Yield sign from a street near your home is pictured to the right. It has the shape of an equilateral triangle with a side length of 2 feet. If the altitude of the triangular sign is drawn, you split the Yield sign in half vertically, creating two 30°-60°-90° right triangles, as shown to the right. For now, we'll focus on the right triangle on the right side. (We could just as easily focus on the right triangle on the left; we just need to pick one.) We know that the hypotenuse is 2 ft., that information is given to us. The shorter leg has length 1 ft. **Why?**

   Verify that the length of the third side, the altitude, is $\sqrt{3}$ ft.



3. The construction manager, Adam, also needs to know the altitude of the smaller triangle within the sign. Each side of this smaller equilateral triangle is 1 ft. long. **Explain why** the altitude of this equilateral triangle is $\sqrt{3}/2$.



4. Now that we have found the altitudes of both equilateral triangles, we look for patterns in the data. Fill in the first two rows of the chart below, and write down any observations you make. Then fill in the third and fourth rows.

Side Length of Equilateral Triangle	**Each 30°- 60°- 90° right triangle formed by drawing altitude**		
	Hypotenuse Length	**Shorter Leg Length**	**Longer Leg Length**
2			
1			
4			
6			

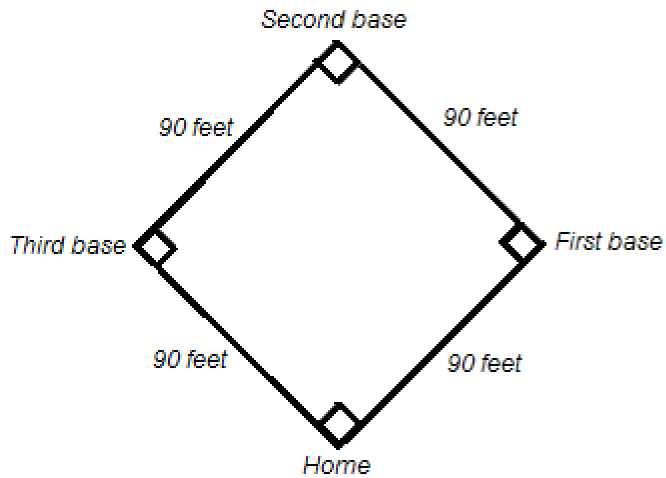
5. What is true about the lengths of the sides of any 30°-60°-90° right triangle? How do you know?

6. Use your answer for *Item 5* as you complete the table below. Do not use a calculator; leave answers exact.

30°-60°-90° triangle	Δ #1	Δ #2	Δ #3	Δ #4	Δ #5	Δ #6	Δ #7	Δ #8
hypotenuse length	11				$3\sqrt{5}$			
shorter leg length		π		$12/5$		$\sqrt{3}$		$\sqrt{2}/2$
longer leg length			$\sqrt{3}/7$				4	

Part 2

A baseball diamond is, geometrically speaking, a square turned sideways. Each side of the diamond measures 90 feet. (See the diagram to the right.) A player is trying to slide into home base, but the ball is all the way at second base. Assuming that the second baseman and catcher are standing in the center of second base and home, respectively, we can calculate how far the second baseman has to throw the ball to get it to the catcher.



7. If we were to split the diamond in half vertically, we would have two 45°- 45°- 90° right triangles. (The line we would use to split the diamond would bisect the 90° angles at home and second base, making two angles equal to 45°, as shown in the baseball diamond to the right below.) Let us examine one of these 45°- 45°- 90° right triangles. You know that the two legs are 90 feet each. Using the Pythagorean Theorem, **verify** that the hypotenuse, or the displacement of the ball, is $90\sqrt{2}$ feet (approximately 127.3 feet) long.

8. Without moving from his position, the catcher reaches out and tags the runner out before he gets to home base. The catcher then throws the ball back to a satisfied pitcher, who at the time happens to be standing at the exact center of the baseball diamond. We can calculate the displacement of the ball for this throw also. Since the pitcher is standing at the center of the field and the catcher is still at home base, the throw will cover half of the distance we just found in *Item 7*. Therefore, the distance for this second throw is $45\sqrt{2}$ feet, half of $90\sqrt{2}$, or approximately 63.6 feet. If we were to complete the triangle between home base, the center of the field, and first base, we would have side lengths of $45\sqrt{2}$ feet, $45\sqrt{2}$ feet, and 90 feet.

   a. Now that we have found the side lengths of two 45°- 45°- 90° triangles, we can observe a pattern in the lengths of sides of all 45°- 45°- 90° right triangles. Using the exact values written using square root expressions, fill in the first two rows of the table at the right.
   b. Show, by direct calculation, that the entries in the second row are related in same way as the entries in the second row.

In each 45°- 45°- 90° right triangle		
Leg Length	Other Leg Length	Hypotenuse Length
90 ft.		
$45\sqrt{2}$ ft.		

9. What is true about the lengths of the sides of any 45°- 45°- 90° right triangle? How do you know?

10. Use your answer for *Item 9* as you complete the table below. Do not use a calculator; leave answers exact.

45°-45°-90° triangle	Δ #1	Δ #2	Δ #3	Δ #4	Δ #5	Δ #6	Δ #7	Δ #8
hypotenuse length			11				$3\sqrt{5}$	
one leg length		π		$\sqrt{2}/2$		$\sqrt{3}$		$12/5$
other leg length	4				$\sqrt{3}/7$			

Finding Right Triangles in Your Environment Learning Task

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

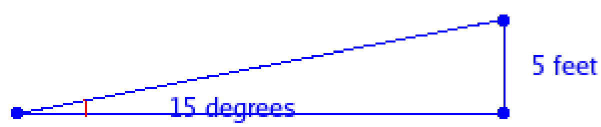
<u>Supplies needed</u>

- Heavy stock, smooth unlined paper for constructing triangles (unlined index cards, white or pastel colors are a good choice)
- Compass and straight edge for constructing triangles
- Protractor for verifying measures of angles
- Ruler in centimeters for measuring sides of constructed triangles

1. Look around you in your room, your school, your neighborhood, or your city. Can you find right triangles in everyday objects? List at least ten right triangles that you find. Draw pictures of at least three of them, labeling the 90° angle that makes the triangle a right triangle.

2. An older building in the school district sits on the side of a hill and is accessible from ground level on both the first and second floors. However, access at the second floor requires use of several stairs. Amanda and Tom have been given the task of designing a ramp so that people who cannot use stairs can get into the building on the second floor level. The rise has to be 5 feet, and the angle of the ramp has to be 15 degrees.

   a. Tom and Amanda need to determine how long that ramp should be. One way to do this is to use a compass and straightedge to construct a 15°- 75°- 90° triangle on your paper. Such a triangle must be similar to the triangle defining the ramp. **Explain why the triangles are similar.**



The diagram is not to scale.

   b. Construct a 15°- 75°- 90° triangle on your paper using straightedge and compass. Use a protractor to verify the angle measurements. (Alternative: If dynamic geometry software is available, the construction and verification of angle measurements can be done using the software.) You'll use this triangle in *part c*.

   c. Use similarity of the ramp triangle and measurements from your constructed triangle to find the length of the ramp. (Save the triangle and its measurements. You'll need them

in another Learning Task also.)

3. Choose one of the types of right triangles that you described in *Item 1* and make up a problem similar to *Item 2* using this type of triangle. Also find an existing right triangle that you can measure; measure the angles and sides of this existing triangle, and then choose numbers for the problem you make up, so that the measurements of the existing triangle can be used to solve the problem. Pay careful attention to the information given in the ramp problem, and be sure to provide, and ask for, the same type of information in your problem.

4. Exchange the triangle problems from *Item 3* among the students in your class.
   a. Each student should get the problem and a sketch of the existing right triangle (along with its measurements) from another student, and then solve the problem from the other student.

   b. For each problem, the person who made up the problem and the person who worked the problem should agree on the solution.

Create Your Own Triangles Learning Task

Supplies needed

- Heavy stock, smooth unlined paper for constructing triangles (unlined index cards, white or pastel colors are a good choice)
- Unlined paper (if students construct triangles in groups and need individual copies)
- Compass and straight edge for constructing triangles
- Protractor for verifying measures of angles
- Ruler in centimeters for measuring sides of constructed triangles

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

1. Using construction paper, compass, straightedge, protractor, and scissors, make and cut out nine right triangles. One right triangle should have an acute angle of 5°, the next should have an acute angle of 10°, and so forth, all the way up to 45°. Note that you should already have a constructed right triangle with an angle of 15° that you saved from the *Finding Right Triangles in the Environment Learning Task.* You can use it or make a new one to have all nine triangles.

   As you make the triangles, you should construct the right angles and, whenever possible, construct the required acute angle. You can use the protractor in creating your best approximation of those angles, such as 5°, for which there is no compass and straightedge construction or use alternate methods involving a marked straightedge.

   As you make your triangles, label both acute angles with their measurements in degrees and label all three sides with their measurement in centimeters to the nearest tenth of a centimeter.

Using what we found to be true about ratios from similar right triangles in the *Circumference of the Earth Task*, we are now ready to define some very important new functions. **For any acute angle in a right triangle, we denote the measure of the angle by θ** and define two numbers related to θ as follows:

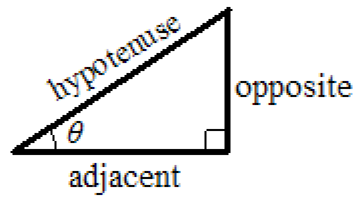
$$\textbf{sine of } \boldsymbol{\theta} = \frac{\text{length of leg opposite the vertex of the angle}}{\text{length of hypotenuse}}$$

$$\textbf{cosine of } \boldsymbol{\theta} = \frac{\text{length of leg adjacent to the vertex of the angle}}{\text{length of hypotenuse}}$$

In the figure at the right below, the terms "opposite," "adjacent," and "hypotenuse" are used as shorthand for the lengths of these sides. Using this shorthand, we can give abbreviated versions of the above definitions:

$$\text{sine of } \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosine of } \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$



2. Using the measurements from the triangles that you created in doing *Item 1* above, for each acute angle listed in the table below, complete the row for that angle. The first three columns refer to the lengths of the sides of the triangle; the last columns are for the sine of the angle and the cosine of the angle. Remember that which side is opposite or adjacent depends on which angle you are considering. (Hint: For angles greater than 45°, try turning your triangles sideways.)
For the last two columns, write your table entries as fractions (proper or improper, as necessary, but no decimals) .

TABLE 1

angle measure	***opposite***	***adjacent***	***hypotenuse***	***sine (opp/hyp)***	***cosine (adj/hyp)***
5°					
10°					
15°					
20°					
25°					
30°					
35°					
40°					
45°					
50°					
55°					
60°					
65°					
70°					
75°					
80°					
85°					

3. Look back at the *Discovering Special Triangles Learning Task, Item 4.*

   a. Use the lengths in the first row of the table from *Item 4* of that learning task to find the values of sine and cosine to complete the **Table 2** below.

TABLE 2.

angle	***sine***	***cosine***
30°		

   b. All right triangles with a 30° angle should give the same values for the sine and cosine ratios as those in **Table 2**. Why?

   c. Do the values for the sine and cosine of a 30° angle that you found for **Table 1** (by using measurements from a constructed triangle) agree with the values you found for the sine and cosine,of a 30° angle in **Table 2**? If they are different, why does this not contradict *part b*?

4. Look back at the *Discovering Special Triangles Learning Task, Item 8.*

   a. Use the table values from *Item 8, part a,* to complete the table below with **exact values** of sine and cosine for an angle of 45°.

TABLE 3

angle	***sine***	***cosine***
45°		

   b. How do the values of sine and cosine that you found for Table 1 compare to the exact values from *part a*? What can you conclude about the accuracy of your construction and measurements?

5. If T is any right triangle with an angle of 80°, approximately what is the ratio of the opposite side to the hypotenuse? Explain.

6. If we changed the measure of the angle in *Item 5* to another acute angle measure, how would your answer change?

7. Explain why the trigonometric ratios of sine and cosine define **functions of θ, where $0° < \theta < 90°$.**

8. Are the functions sine and cosine linear functions? Why or why not?

Discovering Trigonometric Ratio Relationships

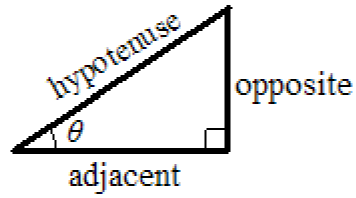
MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

Now that you have explored the trigonometric ratios and understand that they are functions which use degree measures of acute angles from right triangles as inputs, we can introduce some notation that makes it easier to work with these values.

We considered these abbreviated versions of the definitions earlier.

$$\text{sine of } \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosine of } \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$



Now, we'll introduce the notation and abbreviate a bit more. In higher mathematics, the following notations are standard.

sine of θ is denoted by $\sin(\theta)$

cosine of θ is denoted by $\cos(\theta)$

1. Refer back Table 1 from the *Create Your Own Triangles Learning Task*.
   Choose any one of the cut-out triangles created in the *Create Your Own Triangles Learning Task*. Identify the pair of complementary angles within the triangle. (Reminder: complementary angles add up to 90°.) Select a second triangle and identify the pair of complementary angles. Is there a set of complementary angles in every right triangle? Explain your reasoning.

2. Use the two triangles you chose in *Item 2* to complete the table below. What relationships among the values do you notice? Do these relationships hold true for all pairs complementary angles in right triangles? Explain your reasoning.

triangle # and angle	triangle #	θ	$\sin(\theta)$	$\cos(\theta)$
1 – smaller angle	1			
1 – larger angle	1			
2 – smaller angle	2			
2 – larger angle	2			

Summarize the relationships you stated in *Item 3.*

a. If θ is the degree measure of an acute angle in a right triangle, what is the measure of its complement?

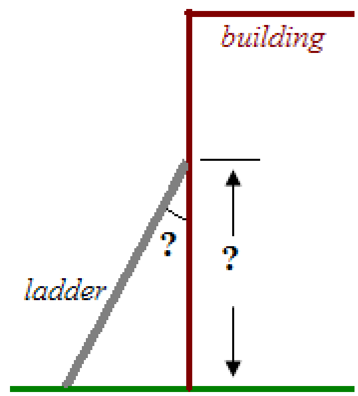
b. State the relationships from *Item 3* as identity equations involving sines, cosines of θ and the measure of its complement. Use the expression from *part a.*

Find That Side or Angle

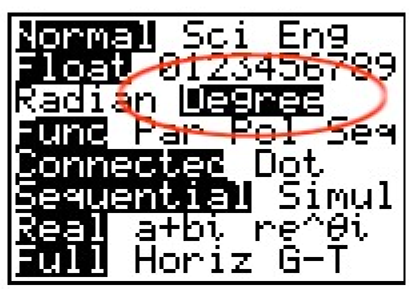
MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Supplies needed
Calculators for finding values of sine and cosine and their inverses

1. A ladder is leaning against the outside wall of a building. The figure at the right shows the view from the end of the building, looking directly at the side of the ladder. The ladder is exactly10 feet long and makes an angle of 60° with the ground. If the ground is level, what angle does the ladder make with the side of the building? How far up the building does the ladder reach (give an exact value and then approximate to the nearest inch)? Hint: Use a known trigonometric ratio in solving this problem.



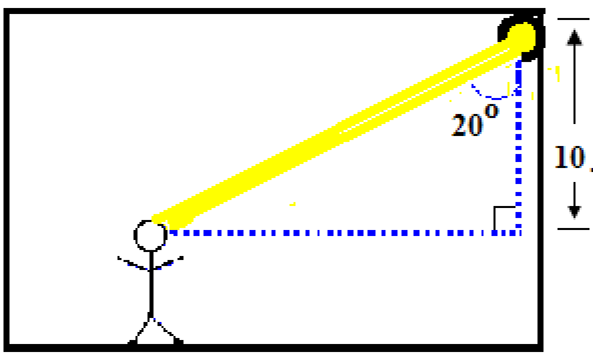
The first problem in this task involves trigonometric ratios in special right triangles, where the values of all the ratios are known exactly. However, there are many applications involving other size angles. Graphing calculators include keys to give values for the sine and cosine functions with very accurate approximations for all trigonometric ratios of degree measures greater than 0° and less than 90°. You should use calculator values for trigonometric functions, as needed, for the remainder of this task.



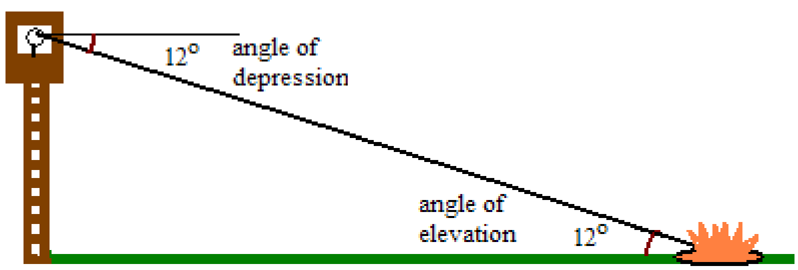
In higher mathematics, it is standard to measure angles in radians. The issue concerns you now because you need to make sure that your calculator is in **degree mode** (and not radian mode) before you use it for finding values of trigonometric ratios. If you are using any of the IT-83/84 calculators, press the MODE button, then use the arrow keys to highlight "Degree" and press enter. The graphic at the right shows how the screen will look when you have selected degree mode. To check that you have the calculator set correctly, check by pressing the TAN

key, 45, and then ENTER. The answer should be 1. If you are using any other type of calculator, find out how to set it in degree mode, do so, and check as suggested above. Once you are sure that your calculator is in degree mode, you are ready to proceed to the remaining items of the question.

2. The main character in a play is delivering a monologue, and the lighting technician needs to shine a spotlight onto the actor's face. The light being directed is attached to a ceiling that is 10 feet above the actor's face. When the spotlight is positioned so that it shines on the actor's face, the light beam makes an angle of 20° with a vertical line down from the spotlight. How far is it from the spotlight to the actor's face? How much further away would the actor be if the spotlight beam made an angle of 32° with the vertical?



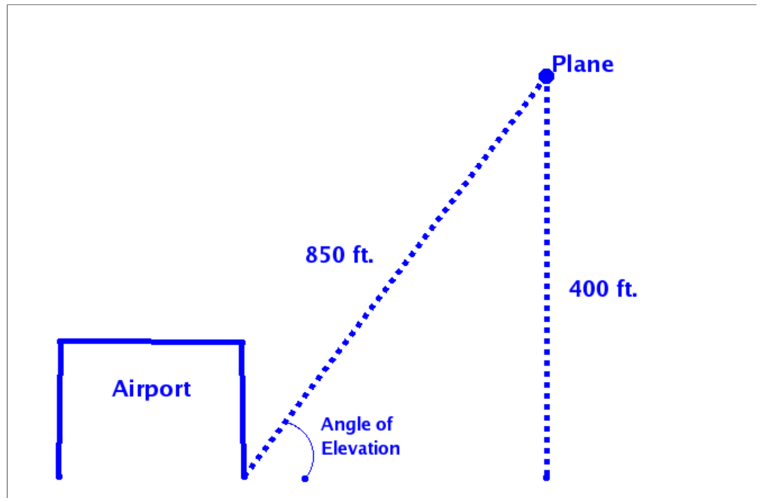
3. A forest ranger is on a fire lookout tower in a national forest. His observation position is 214.7 feet above the ground when he spots an illegal campfire. The angle of depression of the line of site to the campfire is 12°. (See the figure below.)



Note that an angle of depression is measured down from the horizontal; in order to look down at something, you need to lower, or depress, the line of sight from the horizontal. We observe that the line of sight makes a transversal across two horizontal lines, one at the level of the viewer (such as the level of the forest ranger), and one at the level of the object being viewed (such as the level of the campfire). Thus, the angle of depression looking down from the fire lookout tower to the campfire, and the angle of elevation is the angle looking up from the campfire to the tower. The type of angle that is used in describing a situation depends on the location of the observer.

The angle of depression is equal to the corresponding angle of elevation. Why?

4. An airport is tracking the path of one of its incoming flights. If the distance to the plane is 850 ft. (from the ground) and the altitude of the plane is 400 ft, then



   a. What is the sine of the angle of elevation from the ground at the airport to the plane (see figure at the right)?

   b. What is the cosine of the angle of elevation?

   c. Now, use your calculator to find the measure of the angle itself. Pressing "2nd" followed by one of the trigonometric function keys finds the degree measure corresponding to a given ratio. Press 2nd, SIN, followed by the sine of the angle from *part a*. What value do you get?

   d. Press 2nd, COS, followed by the cosine of the angle from *part b*. What value do you get?

Did you notice that, for each of the calculations in *parts c-d*, the name of the trigonometric ratio is written with an exponent of -1? These expressions are used to indicate that we are starting with a trigonometric ratio (sine and cosine,) and going backwards to find the angle that gives that ratio. You'll learn more about this notation later. For now, just remember that it signals that you are going backwards from a ratio to the angle that gives the ratio.

   e. Why did you get the same answer each time?

   f. To the nearest hundredth of a degree, what is the measure of the angle of elevation?

   g. Look back at Table 1 from the *Create Your Own Triangles Learning Task*. Is your answer to *part g* consistent with the table entries for sine and cosine?

5. The top of a billboard is 40 feet above the ground. What is the angle of elevation of the sun when the billboard casts a 30-foot shadow on level ground?

6. An observer in a lighthouse sees a sailboat out at sea. The angle of depression from the observer to the sailboat is 6°. The base of the lighthouse is 50 feet above sea level and the observer's viewing level is 84 feet above the base. (See the figure at the right, which is not to scale.)

   What is the distance from the sailboat to the observer?